**Duct Acoustics**

- **Plane wave**
  - A sound propagation in pipes with different cross-sectional area
  - If the wavelength of sound is large in comparison with the diameter of the pipe the sound propagates as an one-dimensional wave \((\lambda >> d \rightarrow 1\text{-d wave})\)

\[
p' = I e^{i \omega (t - x/c)} + R e^{i \omega (t + x/c)} \quad \text{in } x < 0
\]

\[
= T e^{i \omega (t - x/c)} \quad \text{in } x > 0
\]

*\(I\): 입사, \(R\): 반사, \(T\): 투과*


**Duct Acoustics**

- The mass flux into the junction must equal the mass flux out

\[ \rho_0 A_1 u_1 = \rho_0 A_2 u_2 \]

- The velocity must equal at both sides of the junction

\[ \frac{A_1}{\rho_0 c} (I - R) = \frac{A_2}{\rho_0 c} T \]

- Energy flux \( \langle \text{in} \rangle = \text{Energy flux} \langle \text{out} \rangle \)

\[ A_1 p_1 u_1 = A_2 p_2 u_2 \]

- The pressure of both sides of junction is continuous

\[ I + R = T \]
Duct Acoustics

- The amplitudes of other wave, $R$ and $T$, are can be solve from above the relations

\[ R = \frac{A_1 - A_2}{A_1 + A_2} I, \quad T = \frac{2A_1}{A_1 + A_2} I \]

- The transmission loss, $L_T$, is symmetric in $A_1$ and $A_2$

\[ L_T = 10 \log_{10} \left( \frac{\text{incident power}}{\text{transmitted power}} \right) = 10 \log_{10} \left( \frac{A_1 I^2}{A_2 T^2} \right) = 10 \log_{10} \left( \frac{(A_1 + A_2)^2}{4A_1A_2} \right) \]
Duct Acoustics

- A single expansion-chamber ‘silencer’
  - The simple muffler that is used in car ‘silencer’ consists of inlet and outlet pipes with cross-sectional area $A_1$, and expansion chamber between them of cross-sectional area $A_2$ and length $l$. 

![Diagram](image)
Duct Acoustics

- The first area change occurs at $x=0$ and the second occurs at $x=l$.

\[ p' = Ie^{i\omega(t-x/c)} + Re^{i\omega(t+x/c)} \quad \text{in} \quad x < 0 \]
\[ = Be^{i\omega(t-x/c)} + Ce^{i\omega(t+x/c)} \quad \text{in} \quad 0 < x < l \]
\[ = Te^{i\omega(t-x/c)} \quad \text{in} \quad l < x \]

- The condition of continuity of mass flux,

\[ A_1(I - R) = A_2(B - C) \quad \text{at} \quad x = 0 \]
\[ A_1Te^{-i\omega l/c} = A_2(Be^{-i\omega l/c} - Ce^{-i\omega l/c}) \quad \text{at} \quad x = l \]

- The condition of continuity of pressure

\[ I + R = B + C \quad \text{at} \quad x = 0 \]
\[ Te^{-i\omega l/c} = Be^{-i\omega l/c} + Ce^{-i\omega l/c} \quad \text{at} \quad x = l \]
Duct Acoustics

- The algebraic equation when solved for $R$ and $T$

$$R = \frac{\left(\frac{A_1}{A_2} - \frac{A_2}{A_1}\right)i\sin \frac{\omega l}{c}}{2\cos \omega l / c + i\left(\frac{A_1}{A_2} + \frac{A_2}{A_1}\right)\sin \frac{\omega l}{c}}$$

$$T = \frac{2le^{i\omega l / c}}{2\cos \omega l / c + i\left(\frac{A_1}{A_2} + \frac{A_2}{A_1}\right)\sin \frac{\omega l}{c}}$$

- However, the simple ‘silencer’ does not reduce the total energy of sound in the system.

$$|R|^2 + |T|^2 = |I|^2$$

- Reducing the acoustic energy of transmitted wave

→ Increasing in the reflected wave

- Sound absorbing material

→ reduce the acoustic energy by converting it into heat or vibration
Duct Acoustics

- The transmission loss, $L_T$, is
  $$L_T = 10 \log_{10} \left( \frac{|T|^2}{|T|^2} \right)$$

  $$L_T = 10 \log_{10} \left[ 1 + \frac{1}{4} \left( \frac{A_1}{A_2} - \frac{A_2}{A_1} \right)^2 \sin \left( \frac{\omega l}{c} \right) \right]$$

- The transmission loss is maximum at frequencies for which
  $$\sin \frac{\omega l}{c} = \pm 1 \quad \text{i.e.} \quad \omega = \frac{\pi c}{2l}, \frac{3\pi c}{2l}, \frac{5\pi c}{2l} \ldots \text{etc}$$

- The effect of expansion ratio $m = \frac{A_2}{A_1}$
Duct Acoustics

Note

- ‘Tuning’ for dominant frequencies of noise
- Theory work for only $\lambda \gg d$ “Low frequency wave only”
- High frequency waves behave like 3-D
- Also, the geometrical shape of the duct is not important (provided the area change occurs in a distance short in comparison with the wavelength
Duct Acoustics

Ref. “Theoretical and experimental investigation of mufflers with comments on engine-exhaust muffler design”, Davis et al. NACA 1192(1954)

Effect of expansion chamber ratio

Effect of expansion chamber shape
Higher order modes

- As an illustration, the sound of frequency $\omega$ in a rigid walled duct of square cross-section with sides of length $a$ is considered

$$p'(x,t) = f(x_1)g(x_2)h(x_3)e^{i\omega t}$$

- With substitution for $p'$ into the wave equation,

$$\frac{f''}{f} = -\frac{g''}{g} - \frac{h''}{h} - \frac{\omega^2}{c^2} = -\alpha^2$$
Higher order modes

• Since a wall boundary condition is applied, function $f$ is derived like this

$$f(x_1) = A_1 \cos\left(\frac{m \pi x_1}{a}\right), \quad \text{for some integer } m$$

• Similarly function $g$ is derived like this

$$g(x_2) = A_2 \cos\left(\frac{n \pi x_1}{a}\right), \quad \text{for some integer } n$$

• Finally, function $h$ is derived to the propagation form

$$h(x_3) = A_{mn} e^{-i k_{mn} x_3} + B_{mn} e^{i k_{mn} x_3} \quad k_{mn} = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2} (m^2 + n^2)}$$

• The axial phase speed, $c_p = \omega / k_{mn}$ is now a function of the mode number and the propagation of a group of waves will cause them to disperse.
Higher order modes

- The pressure perturbation in the \((m,n)\) mode has the form

\[
p'(x,t) = \cos \left( \frac{m \pi x_1}{a} \right) \cos \left( \frac{n \pi x_2}{a} \right) \left[ A_{mn} e^{-ik_{mn}x_3} + B_{mn} e^{ik_{mn}x_3} \right] e^{i\omega t}
\]

- When \(k_{mn}\) is real, the pressure perturbation equation represents that waves are propagating down the \(x_3\) axis with phase speed.

- When \(k_{mn}\) is purely imaginary, i.e. exceeds the cut-off frequency, the strength of mode varies exponentially with distance along the pipe. Such disturbances are evanescent.
Pipes of varying cross-section

Wave equation

- If the pipe diameter is small in comparison with both the acoustic wavelength and the length scale over which the cross-sectional area change, most particle motions are longitudinal.

- Conservation of mass
  \[ A \frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial}{\partial x} (uA) \]

- Linearized momentum equation is
  \[ \rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p'}{\partial x} \]

- Modified wave equation
  \[ \frac{A}{c^2} \frac{\partial^2 p'}{\partial t^2} = \frac{\partial}{\partial x} \left( A \frac{\partial p'}{\partial x} \right) \]
Pipes of varying cross-section

Application to the ‘exponential horn’

- Evaluation of the case of ‘exponential horn’ which cross-sectional area defined as, \( A(x) = A_0 e^{\alpha x} \)

- For such an area variation of wave equation simplifies to

\[
\frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = \frac{\partial^2 p'}{\partial x^2} + \alpha \frac{\partial p'}{\partial x}
\]

- The pressure perturbation in sound waves of frequency \( \omega \) then has the form

\[
p'(x,t) = e^{-\alpha x/2} \left\{ A e^{i(\omega x - kx)} + B e^{i(\omega x + kx)} \right\}
\]

- Disturbance with \( \omega > \alpha c/2 \) propagates and the pressure but not the energy flux attenuates during propagation, while lower frequency modes are ‘cut-off’
Normal transmission

Physics at the interface

- When a sound wave crosses an interface between two different fluids some of the acoustic energy is usually reflected.

\[ I e^{i\omega(x/c_0)} \quad R e^{i\omega(x/c_0)} \quad T e^{i\omega(x/c_1)} \]

- There are two boundary conditions
  - The pressure on the two sides of the boundary must be equal
  - The particle velocities normal to the interface must be equal

\[ \lambda = cT = \frac{c}{f} = \frac{2\pi c}{\omega} \quad \lambda_0 = \frac{2\pi c_0}{\omega} \quad \lambda_1 = \frac{2\pi c_1}{\omega} \]
**Normal transmission**

- The pressure must be equal at the interface: \( I + R = T \)
- The particle velocities normal to the interface must be equal

\[
\frac{I}{\rho_0 c_0} - \frac{R}{\rho_0 c_0} = \frac{T}{\rho_1 c_1}
\]

- The result pressure coefficients, \( R \) and \( T \), are determined with \( I \)

\[
R = \left\{ \frac{\rho_1 c_1 - \rho_0 c_0}{\rho_1 c_1 + \rho_0 c_0} \right\} I \\
T = \left\{ \frac{2\rho_1 c_1}{\rho_1 c_1 + \rho_0 c_0} \right\} I
\]

- Velocity Transmission Coefficient: 

\[
\frac{T / \rho_1 c_1}{I / \rho_0 c_0} = \frac{2\rho_0 c_0}{\rho_1 c_1 + \rho_0 c_0}
\]

- The energy flux of the incident wave per unit cross sectional area is equal to that of the reflected and transmitted waves

\[
\frac{R^2}{\rho_0 c_0 \rho_1 c_1} + \frac{T^2}{\rho_1 c_1} = \frac{(\rho_1 c_1 - \rho_0 c_0)^2 I^2}{(\rho_1 c_1 + \rho_0 c_0)^2 \rho_0 c_0} + \frac{4\rho_1^2 c_1^2 I^2}{(\rho_1 c_1 + \rho_0 c_0)^2 \rho_1 c_1} = \frac{I^2}{\rho_0 c_0}
\]
Normal transmission

Reflection from a high and low impedance fluid

- A typical example is aerial sound waves incident onto a water surface. \((\rho_0 c_0 \ll \rho_1 c_1)\)

\[
R = I, \quad T = 2I
\]

- Velocity transmission coefficient

\[
\frac{2\rho_0 c_0}{\rho_1 c_1 + \rho_0 c_0} \approx 0
\]

so, the transmission wave carries negligible energy
• **Normal transmission**

- **Reflection from a high and low impedance fluid**

  ![Diagram showing transmission and reflection of sound waves between air and water]

  - In the opposite case, for sound in water incident onto a free surface with air, the reflected and transmitted waves are
  
  \[ R = -I , \quad T = 0 \]

  - The acoustic energy is **totally reflected**
Sound propagation through walls

Effect of a wall in transmission

- A sound wave normally incident on a plane material layer partitioning a fluid which has uniform acoustic properties, $\rho_0 c_0$
- Some sound will be reflected from the layer and some will be transmitted through the wall

\[ I e^{i\omega(t-x/c_0)} \quad R e^{i\omega(t+x/c_0)} \quad T e^{i\omega(t-x/c_0)} \]

\[ \rho_0, c_0 \quad \text{Incompressible material layer} \quad \rho_0, c_0 \]
Sound propagation through walls

- There are two boundary conditions that must be satisfied at all times and points
  - The velocity of the wall must be equal to wave of each side
  - A pressure difference across the wall in order to provide the force necessary to accelerate unit area of the surface of material
- By continuity the velocity of wall,
  \[ u = (I - R) \frac{e^{i\omega t}}{\rho_0 c_0} = \frac{T}{\rho_0 c_0} e^{i\omega t} \]
- The pressure difference is the net force of mass per unit area of the wall
  \[ p_1' = (I + R)e^{i\omega t} \]
  \[ p_2' = Te^{i\omega t} \]
  \[ (I + R - T)e^{i\omega t} = m \frac{\partial u}{\partial t} = \frac{i \omega T}{\rho_0 c_0} e^{i\omega t} \]
Sound propagation through walls

The result pressure coefficients, $R$ and $T$, are determined with

$$R = \left( \frac{i \omega m}{2 \rho_0 c_0 + i \omega m} \right) I$$

$$T = \left( \frac{2 \rho_0 c_0}{2 \rho_0 c_0 + i \omega m} \right) I$$

Surface Impedance

$$\frac{p_1'}{u} = \rho_0 c_0 \frac{(I + R)}{(I - R)} = \rho_0 c_0 \frac{1 + i \omega m}{1 - i \omega m}$$

$$\frac{p_2'}{u} = \rho_0 c_0 \frac{(I + R)}{T} = \rho_0 c_0 + i \omega m$$

Energy transmitted

$$\frac{|T|^2}{\rho_0 c_0} = \frac{4 \rho_0^2 c_0^2}{4 \rho_0^2 c_0^2 + \omega^2 m^2} \frac{I^2}{\rho_0 c_0}$$
Sound propagation through walls

- The transmission loss is dependent on the frequency $\omega$.

\[ L_T = 10 \log_{10} \left( \frac{4 \rho_0^2 c_0^2}{4 \rho_0^2 c_0^2 + \omega^2 m^2} \right)^{-1} \]

- For high frequency ($\omega m \gg \rho_0 c_0$), the sound waves mostly reflected

- For low frequency ($\omega m \ll \rho_0 c_0$), the sound waves mostly travels through the wall with very little attenuation

- “Low frequency waves get through a massive wall easily, while high frequency waves are effectively stopped”
Sound propagation through walls

Example) Attenuation by a wall

- Transmission loss

\[ L_T = 10 \log_{10} \left( \frac{|I|^2}{|T|^2} \right) \]
\[ L_T = 10 \log_{10} \left( \frac{4 \rho_0^2 c_0^2}{4 \rho_0^2 c_0^2 + \omega^2 m^2} \right)^{-1} \]

For \( f = 10 \text{Hz} \):
\[ L_T \approx 12 \text{dB} \]

For \( f = 1 \text{kHz} \):
\[ L_T \approx 50 \text{dB} \]